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Inverted Cauchy problem for the Laplace equation in engineering design

R. H. NILSON and Y. G. TSUEI

Department of Mechanical Engineering, University of Cincinnati, Cincinnati, Ohio 45221, USA

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SUMMARY
A Cauchy problem for the Laplace equation is solved by analytic continuation of the space variables on the plane of the complex potential, thereby obtaining an explicit expression for the geometry of physical boundaries of interest. In an illustrative application to the inverse free boundary problem of electrochemical machining, the general solution comprises a closed-form description of a tool family which can be used to machine a prescribed workpiece. The method is extended to include the effects of variable electrolyte conductivity, and a general tool design procedure is suggested in which an analytic series with correct asymptotic behavior is used to represent the given workpiece geometry. Applications in other fields such as heat conduction and hydrodynamics are discussed. The inverted formulation described herein affords considerable advantage and generality in solving Cauchy problems which are encountered in engineering design.

1. Introduction

Free boundary problems for the Laplace equation which arise in ideal fluid flow, heat conduction, and electrochemical machining are characterized by a boundary surface Γ_0 along which both Dirichlet and Neumann boundary conditions must be satisfied. In the *direct* free boundary problem, the position of the free boundary Γ_0 is not prescribed but must be determined in accordance with the given location of fixed boundaries; whereas in the *inverse* free boundary problem, the position of Γ_0 is prescribed and the problem consists in locating the alternative positions for the fixed boundaries of the field. Depending upon the nature of the application, the fixed boundaries of interest may be either a family of equipotential lines Γ_ϕ or a family of flux lines Γ_ψ. Although the literature of inverse free boundary problems is much less extensive than that of the related direct problems, it is believed that in many cases the inverse problem is of greater practical interest since it comprises a direct approach to engineering design problems.

The present investigation is concerned with the inverse free boundary problem of potential theory. Since this is recognized as a Cauchy problem for the Laplace equation, a solution can be found by analytic continuation of a function of a complex variable. A novel feature of the present method is analytic continuation of the space variables in the plane of the complex potential such that the fixed boundaries of interest are obtained explicitly. It is emphasized that in previous formulations of the problem, the fixed boundaries of interest can only be located by numerical search procedures or by finding the inverse of a complex function. Although the present procedure is applicable to free boundary problems and design problems of fluid flow and heat transfer, the inverse problem of electrochemical machining (ECM) is selected for the purpose of demonstration. It is found that the method described herein provides a direct means for determining the family of tool shapes which can be used to machine a desired workpiece geometry.

A formulation of the inverse problem of ECM is followed by discussion of a solution by Krylov [9]. It is shown that it is preferable to reformulate the Cauchy problem on the plane of the complex potential. A very convenient general solution is obtained which serves as the basis of a suggested design procedure for ECM tools. It is found that the present method is easily extended to include the effects of variable electrolyte conductivity. Applications of the method to other free boundary and engineering design problems is discussed.

2. Formulation of the inverse problem of ECM

Electrochemical machining is a process of metal erosion in which the anode is a workpiece from which metal is removed and the cathode is a shaped tool. An electric potential is applied across the gap between the tool and workpiece, and the electrolyte is pumped through the gap to remove the products of erosion. In most applications, a high rate of metal removal is maintained by feeding the tool toward the workpiece during the machining process, and a "steady" state is reached in which the worksurface maintains a fixed position in the reference frame of the moving tool. The steady-state worksurface resembles the tool shape, but is not precisely congruent. At high feedrates, the worksurface geometry is influenced by variations in electrolyte conductivity which result from variation in both the electrolyte temperature and the void fraction of liberated hydrogen gas.

The literature of ECM is primarily devoted to experimental studies and mathematical models of the transport process which are based upon parallel current approximations such as those of Thorpe and Zerkle [14] or Loutrel and Cook [10]. The direct free boundary problem for the Laplace equation has been solved through conformal mapping (assuming constant conductivity) for a few special cases by Collett, Hewson-Browne, and Windle [1] and by Dietz, Gunther, and Otto [2], whereas Tipton [15] describes a finite difference procedure, and Kawafune, Mikoshiba, and Noto [8] suggest constructing an analogue model. The so-called cosine law of machining which appears extensively in the literature is a first order approximation for the inverse free boundary problem which has been solved exactly by Krylov for some mathematically tractable geometries. The present discussion describes a general method for solving the inverse free boundary problem which is even applicable to cases with variable electrolyte conductivity.

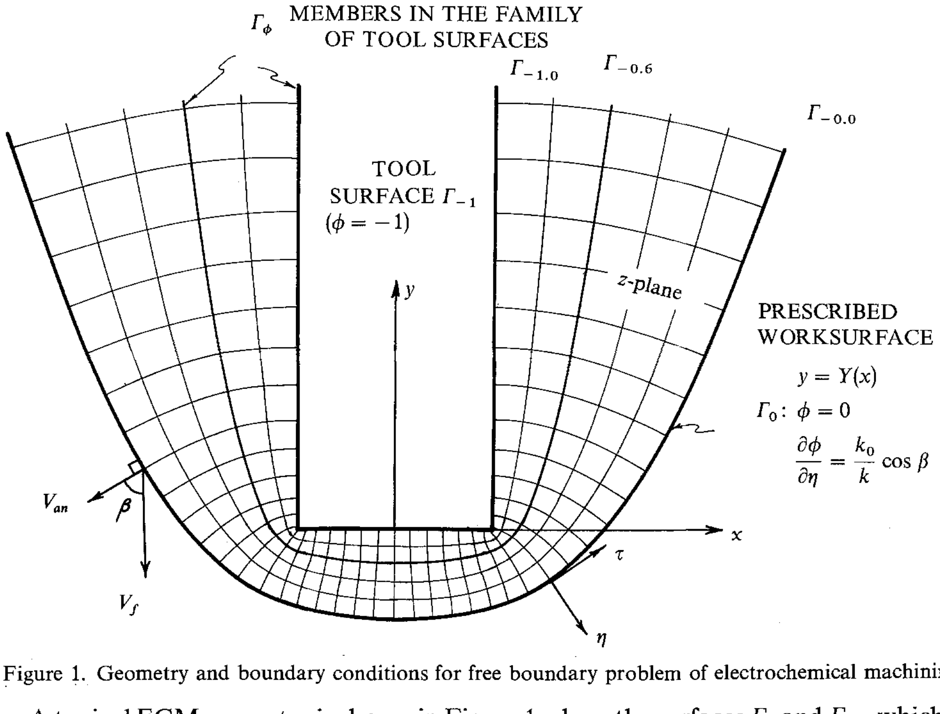


Figure 1. Geometry and boundary conditions for free boundary problem of electrochemical machining.

A typical ECM geometry is shown in Figure 1 where the surfaces Γ_0 and Γ_{-1} which bound the workpiece and tool respectively are separated by an electrolyte filled gap. Although Γ_0 and Γ_{-1} may be three-dimensional, the present discussion is limited to two-dimensional geometry. The shape of the leading surface of the tool may vary, but the sides of the tool usually become almost parallel to the feed direction at some distance back from the frontal gap and are sometimes partially insulated.

The metal erosion process which occurs on the worksurface Γ_0 is controlled by the current density distribution. Since the current flux is everywhere normal to Γ_0, the mass rate of metal removal is given by

$$\dot{m}_{an} = \lambda_a k \frac{\partial \Phi}{\partial N} \tag{1}$$

where Φ is the electric potential, N is the outer normal of Γ_0, k is the electrolyte conductivity, and λ_a is the electrochemical equivalent of the anode material. Thus, the erosion process produces a locally normal absolute velocity V_{an} of Γ_0 as given by

$$V_{an} = \frac{\lambda_a k}{\rho_a} \frac{\partial \Phi}{\partial N} \tag{2}$$

where ρ_a is the density of the anode material. The relative velocity of Γ_0 with respect to the moving tool is obtained by vector addition of the normal cutting velocity V_{an} and the feed velocity which is equal in magnitude to the feedrate V_f but in the opposite direction. In steady state machining the relative velocity of Γ_0 with respect to the tool must be everywhere tangent to Γ_0. Thus, it is necessary that $V_{an} = V_f \cos \beta$ or equivalently

$$\frac{\partial \Phi}{\partial N} \frac{k\lambda_a}{\rho_a V_f} = \cos \beta \tag{3}$$

where β is the angle between the outer normal of Γ_0 and the feed direction.

In a steady state ECM process such as that of Fig. 1 the electric potential must satisfy the following partial differential equation

$$\frac{\partial}{\partial x}\left(k \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial \phi}{\partial y}\right) = 0 \tag{4}$$

and the boundary conditions

$$\begin{aligned} &\phi = -1 \quad \text{on} \quad \Gamma_{-1} \\ &\phi = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial \eta} = \frac{k_0}{k} \cos \beta \text{ on } \Gamma_0 \end{aligned} \tag{5}$$

where $\phi = \Phi/\Delta\Phi$, $\Delta\Phi$ is the potential difference between the electrodes, and the cartesian coordinates x and y and the normal coordinate η have been non-dimensionalized with respect to the nominal frontal gap

$$g_0 = \frac{\lambda_a k_0 \Delta\Phi}{V_f \rho_a}. \tag{6}$$

There are two different problems to be distinguished:

(1) The direct problem in which the tool shape Γ_{-1} is given and the unique worksurface Γ_0 is to be determined.

(2) The inverse problem in which the worksurface Γ_0 is given and the family of tool shapes Γ_ϕ is to be determined. To illustrate the infinite multiplicity of solutions for the inverse problem, suppose that the feedrate is moderate, and the electrolyte velocity is high enough to ensure a uniform electrolyte conductivity k throughout the gap. For the given worksurface Γ_0 any one of the equipotential lines Γ_ϕ which lies in the gap between Γ_0 and Γ_{-1} represents a prospective tool shape which can be used to produce the worksurface Γ_0. For example, the tool shape $\Gamma_{-0.5}$ may be used to ECM the surface Γ_0 at the same feedrate but with about half as much applied voltage as is necessary for the original tool shape Γ_{-1}. The smaller the gap between Γ_ϕ and Γ_0, the smaller will be both the required potential difference $\Delta\Phi$ and the power consumed, but for the same feedrate each of the tools Γ_ϕ will produce the same steady state current density (3) on Γ_0. Of course, with a very small gap it would be difficult both to maintain a high electrolyte flow velocity and to avoid accidental contact of the electrodes.

The present discussion concerns the inverse problem of two-dimensional ECM wherein the geometry of the worksurface is prescribed in the form

$$y = Y(x) \text{ on } \Gamma_0 . \tag{7}$$

Assuming for the moment that the electrolyte conductivity is uniform, the potential function is harmonic and must satisfy both of the conditions

$$\phi = 0 \text{ and } \frac{\partial \phi}{\partial \eta} = \cos \beta \text{ on } \Gamma_0 \tag{8}$$

and the problem consists in finding the location of the equipotential lines Γ_ϕ which comprise the admissable tool family. It is noted that the following functions are also available from the prescribed data (7) and (8):

$$h_1 = \frac{\partial \phi}{\partial \eta}(\tau) \quad \text{on} \quad \Gamma_0 \tag{9}$$

$$h_2 = x(\tau) \quad \text{on} \quad \Gamma_0 \tag{10}$$

$$h_3 = \frac{\partial x}{\partial \eta}(\tau) \quad \text{on} \quad \Gamma_0 \tag{11}$$

where τ is an arc length parameter on Γ_0.

Since the electric potential ϕ satisfies the Laplace equation, the complex potential $w = \phi + i\psi$ is an analytic function of $z = x + iy$ in the physical z-plane. Additionally, the auxiliary complex variable $\zeta = \tau + i\eta$ is defined as shown in Figure 1 such that Γ_0 lies along the real axis of the ζ-plane where it is parameterized by τ which represents arc length in the physical plane.

Both of the solution methods which are subsequently discussed make use of analytic continuation. Suppose that $\omega = \mu + i\nu$ is an analytic function of the complex variable $\gamma = t + is$ and that

$$\mu = f(t) \text{ and } -\frac{\partial \mu}{\partial s} = g(t) \tag{12}$$

on the real axis $\mathrm{Im}(\gamma) = 0$ where $f(t)$ and $g(t)$ are given analytic functions. Then, by integration of a Cauchy–Riemann condition

$$\nu = \int_0^t g(t) dt \tag{13}$$

also on $\mathrm{Im}(\gamma) = 0$. Thus, the prescribed data (12) is sufficient to determine ω within its region of existance where it may be expressed as

$$\omega(\gamma) = f(\gamma) + i \int_0^\gamma g(\gamma) d\gamma \tag{14}$$

which by inspection agrees with the data specified on the real axis. It is noted that in (12), f and g are real valued functions of a real variable, whereas in (14) they are complex valued functions of a complex variable.

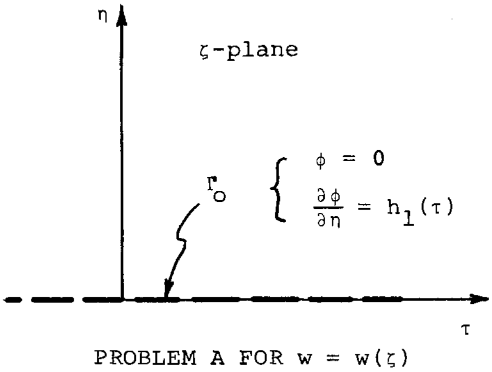
3. Analytic continuation on an auxiliary plane

Krylov presents a general solution of the inverse problem of ECM by analytic continuation on the plane of a complex parameter, that is, in the form $w = w(\zeta)$ and $z = z(\zeta)$ where $\zeta = \tau + i\eta$ is the auxiliary complex variable previously defined. This approach involves the solution of the two distinct Cauchy problems shown on the ζ-plane in Figure 2. First consider problem A wherein $w(\zeta)$ corresponds to $\omega(\gamma)$ in (14) and the prescribed data are

$$\phi = 0 \text{ and } \frac{\partial \phi}{\partial \eta} = h_1(\tau) \text{ on } \Gamma_0 . \tag{15}$$

So in this case, the functions corresponding to f and g in (12) are

$$f(\tau) = 0 \text{ and } g(\tau) = -h_1(\tau) \tag{16}$$



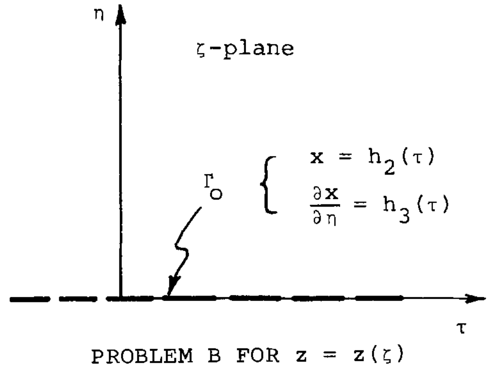


Figure 2. Cauchy problems on the auxiliary plane.

on the real axis $\text{Im}(\zeta)=0$, and the solution of problem A is then given by

$$w(\zeta) = 0 - \text{i}\int_0^\zeta h_1(\zeta)d\zeta\,. \tag{17}$$

Now consider problem B wherein $z(\zeta)$ corresponds to $\omega(\gamma)$ in (14) and the prescribed data are

$$x = h_2(\tau) \text{ and } \frac{\partial x}{\partial \eta} = h_3(\tau) \text{ on } \Gamma_0 \tag{18}$$

or, equivalently

$$f(\tau) = h_2(\tau) \text{ and } g(\tau) = -h_3(\tau) \tag{19}$$

on the real axis $\text{Im}(\zeta)=0$. Then the solution of problem B is

$$z(\zeta) = h_2(\zeta) - \text{i}\int_0^\zeta h_3(\zeta)d\zeta\,. \tag{20}$$

By this procedure, the solution is obtained in the form $w=w(\zeta)$ and $z=z(\zeta)$. It is now necessary to find the location in the z-plane of the equipotential curves Γ_ϕ. Since $z=z(\zeta)$ is given by (20), it is first required that these curves be located in the ζ-plane. Only in some very simple cases such as those discussed by Krylov is it possible to find an analytic expression for $\zeta=\zeta(w)$ by inversion of the relationship (17) for $w=w(\zeta)$. In general it is difficult to locate the equipotential curves in the ζ-plane and a numerical search procedure is required.

4. Inverted Cauchy problem by analytic continuation

The present approach to the inverse free boundary is based upon a property of harmonic functions which has been used previously, for example by Thom and Apelt [13], to solve boundary value problems. If ϕ and ψ are conjugate functions in the z-plane, then x and y are conjugate functions in the w-plane. Since the equipotential free boundary Γ_0 is parameterized by ψ, it is necessary to state the inverted Cauchy problem in the plane of the complex variable $w^*=\psi+\text{i}\phi$ as indicated in Figure 3, and because the prescribed curve Γ_0 is most naturally given as $y=Y(x)$, it is very convenient to work with the dependent variable $z^*=y+\text{i}x$. By differentiating $z^*[\zeta(w^*)]$,

$$\left.\frac{\partial y}{\partial \phi}\right|_\psi = \left.\frac{\partial y}{\partial \eta}\right|_\tau \left.\frac{\partial \eta}{\partial \phi}\right|_\psi + \left.\frac{\partial y}{\partial \tau}\right|_\eta \left.\frac{\partial \tau}{\partial \phi}\right|_\psi \tag{21}$$

and in view of (8), and the Cauchy–Riemann conditions it is obtained that

$$\frac{\partial y}{\partial \phi} = -1 \text{ and } \frac{\partial x}{\partial \psi} = 1 \text{ on } \Gamma_0 \tag{22}$$

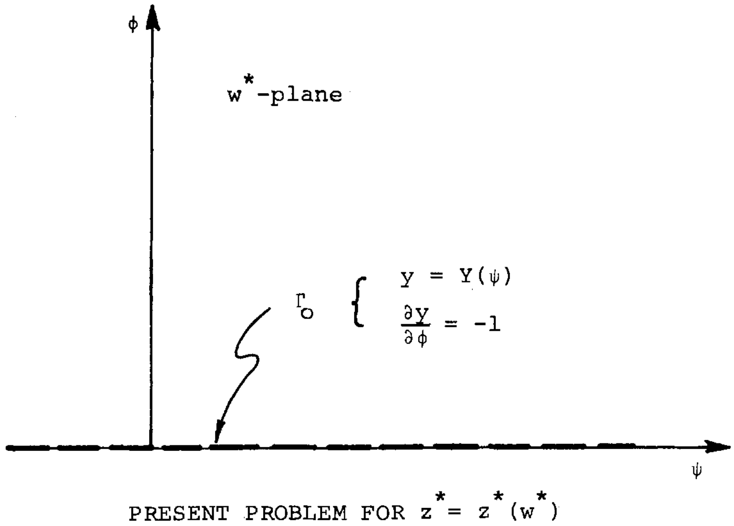


Figure 3. Cauchy problem on the inverted plane.

whereby, without loss of generality

$$x = \psi \text{ on } \Gamma_0 . \tag{23}$$

Thus, the Cauchy problem for $z^*(w^*)$ is defined by

$$y = Y(\psi) \text{ and } \frac{\partial y}{\partial \phi} = -1 \text{ on } \Gamma_0 \tag{24}$$

or, by the notation of (12),

$$f(\psi) = Y(\psi) \text{ and } g(\psi) = 1 \tag{25}$$

on the real axis $\text{Im}(w^*) = 0$, and the solution is obtained from (14) as

$$z^*(w^*) = Y(w^*) + i\psi - \phi . \tag{26}$$

Thus, the equipotential curves Γ_ϕ are obtained explicitly by simply selecting a value for ϕ and mapping out x and y as functions of the parameter ψ. For example, if the free boundary Γ_0 is prescribed as

$$Y(x) = \tfrac{1}{2}(x^2 - 1) \tag{27}$$

the solution becomes

$$y + ix = [\psi^2 - (\phi+1)^2]/2 + i\psi(\phi+1) \tag{28}$$

which describes the family of parabolic tool shapes shown in Figure 4. Any one of the equipotential curves which lies above Γ_0 (including the semi-infinite flat plate along the y-axis) can be used as a tool shape for machining the concave parabolic surface Γ_0, whereas by reversing the polarity on the equipotentials which lie below Γ_0 it is possible to produce a convex work-surface of the same shape.

By reformulating the Cauchy problem on the inverted plane, the family of tool shapes is obtained explicitly without any need for numerical search or finding the inverse of an analytic function. This is one of the features of the present method which is advantageous in practical tool design applications.

5. Application to ECM tool design

A general procedure for ECM tool design is based upon the above solution (26) of the inverse

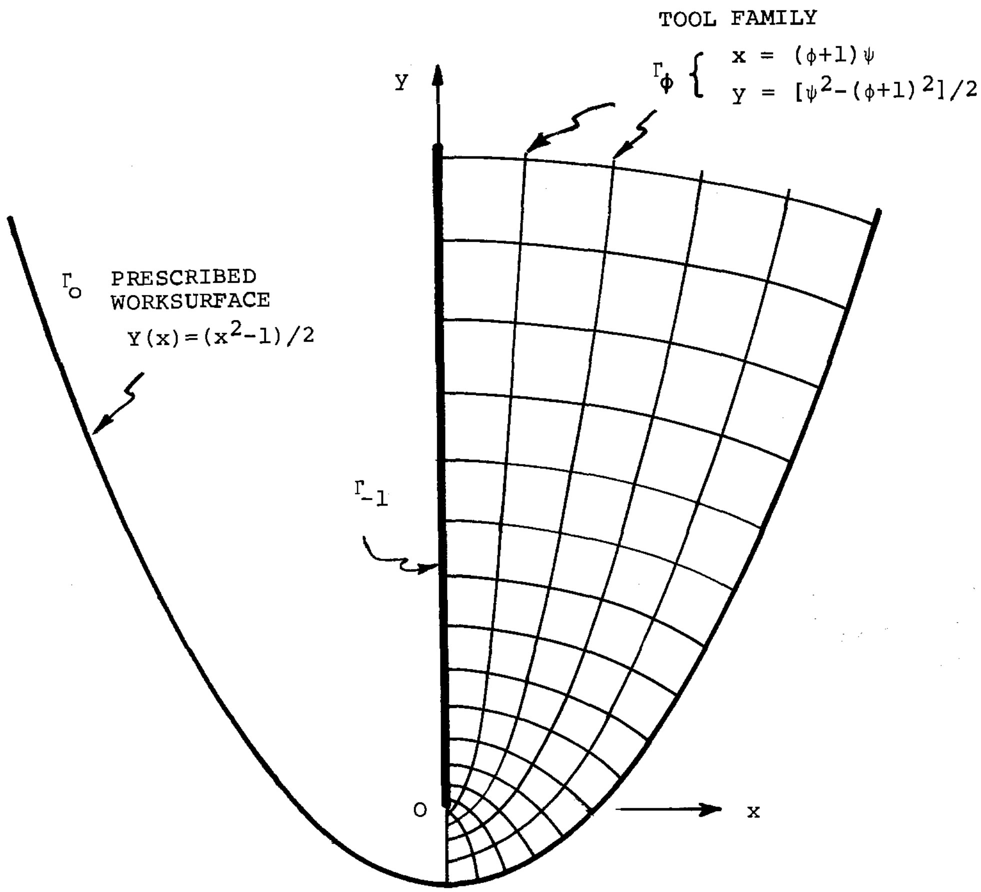


Figure 4. Tool family for worksurface prescribed as $Y(x) = (x^2 - 1)/2$.

problem for $z^*(w^*)$. Although the desired workpiece geometry $Y(x)$ is quite arbitrary, there are several considerations which restrict the class of admissible curves.

a. Only analytic worksurfaces can be produced by ECM, that is, the process will not reproduce discontinuities in derivatives of the tool surface.

b. The equipotential curves of the solution should include tool shapes which have practical application in a conventional ECM process. For example, if the worksurface Γ_0 is to be machined by an asymptotically straight–sided and equipotential tool, y must increase asymptotically with $x^2/2$ on Γ_0; or if the tool is to be partially insulated, Γ_0 must be asymptotically parallel to the feed direction. Thus, the prescription of Γ_0 should be compatible with the asymptotic behavior produced by some reasonable tool shape.

c. For smooth Γ_0, the equipotential curves which lie near Γ_0 can usually be interpreted as acceptable tool shapes, but the curves which lie far away may be cusped or looped as discussed by Krylov. This behavior is not a consequence of the present method but is characteristic of the Cauchy problem for the Laplace equation as discussed by Garabedian [5].

d. In engineering applications, the worksurface is prescribed by data points or connected line segments which may not conform to the above requirements for smoothness and asymptotic behavior. Even if the analytic continuation were to be undertaken by finite difference methods such as those described by Garabedian and Lieberstein [7] or Frank [4], it would be best to begin by constructing an analytical approximation of the prescribed data which meets these requirements.

In view of the above considerations, it is appropriate to pursue a tool design approach which replaces the arbitrarily prescribed worksurface $Y(x)$ with a series approximation $\tilde{Y}(x)$ by analytic functions. For example, the series

$$\tilde{Y}(x) = a_0 + \tfrac{1}{2}x^2 + \sum_{n=1}^{N} \sum_{m=1}^{M} a_{nm} x^{(n-1)} e^{-mx} \tag{29}$$

is sufficiently general to represent a wide variety of worksurfaces which can be machined by equipotential tools which are asymptotically straight-sided. For a desired worksurface $Y(x)$, the coefficients of (29) can be determined by least squares regression, then the resulting analytical approximation is substituted into the general solution (26), and the location of the appropriate tool shapes Γ_ϕ is found by direct numerical computation. Preliminary test calculations indicate that this procedure provides very good results for tool shapes such as those of Figure 1.

Thus, the tool design problem becomes one of selecting an approximating series which is sufficiently general and has the correct asymptotic behavior for a particular application. It is clear that the procedure outlined above can be easily programmed for a digital computer and that the simplicity of the general solution $z^* = z^*(w^*)$ prevents computational difficulties which would otherwise be encountered.

6. The inverse problem with variable electrolyte conductivity

At high feed rates, the ECM gap geometry is influenced by the variation in effective electrolyte conductivity which results from variations in both the void fraction of hydrogen gas and the temperature of the electrolyte. In such cases, a very adequate approximation is provided by a one-dimensional two-phase flow model such as that of Fluerenbrock, Zerkle, and Thorpe [3] which accounts for property variations in the direction of electrolyte flow, or equivalently in the ψ direction of the w^*-plane. Since the liquid mass flow rate $\dot{m}_f$ is constant along the gap, both the bulk temperature T_b and the mass fraction of hydrogen gas $\alpha = \dot{m}_g/\dot{m}_f$ increase linearly with the current parameter ψ. Furthermore, empirical relations express the effective electrolyte conductivity as a function of T_b and the volume fraction of hydrogen gas $\alpha' = \alpha v_g/v_f$ where v_g and v_f are the specific volumes of each phase; that is, $k = k(T_b, \alpha')$, and $\alpha' = \alpha'(\alpha, T_b, p)$ where p is the pressure. Thus, if the pressure distribution is given, the effective conductivity k becomes an explicit function of ψ. Using the physical model formulated by Thorpe and Zerkle [14] and assuming constant pressure, it is found that

$$\frac{k}{k_0} = 1 + C_1\psi + C_2\psi^2 + C_3\psi^3 + \ldots \tag{30}$$

where the coefficients C's depend upon physical constants and process parameters which are all independent of the geometry of the gap.

With k/k_0 being a function of ψ only, it is easy to show that ϕ still satisfies the Laplace equation as does its conjugate ψ. Hence, the inverse problem statement is the same as before except that $x = X(\psi)$ on Γ_0 where

$$X(\psi) = \int_0^{\psi} \frac{k}{k_0}(\psi)\,d\psi \quad \text{on } \Gamma_0 \tag{31}$$

and since $y = Y(x)$ on Γ_0, or equivalently

$$y = Y[X(\psi)] \quad \text{on } \Gamma_0 \tag{32}$$

the solution is given by

$$z^* = Y[X(w^*)] + iX(w^*) \tag{33}$$

which again poses no computational difficulties. Thus, by formulating the inverse problem on the w^*-plane, the difficulties usually associated with variable conductivity become much more tractable.

7. Other applications

Inverse free boundary problems arise not only in connection with direct free boundary problems, but in more general applications as well. Since inverse problems consist in finding the location of an unknown field boundary (Γ_ϕ or Γ_ψ) along which only one boundary condition is to be maintained, they are usually encountered in a design context. Thus, the present inverted formulation of a Cauchy problem is particularly suitable for a variety of engineering design problems.

Inverse problems of conduction heat transfer are characterized by a given boundary along which both the temperature and the heat flux distribution are specified. For example, the problems of porous cooling and radiation melting described respectively by Goldstein and Siegel [7] and Siegel [12] have the same boundary conditions on the free surface as the preceding application to ECM, whereas in the steady state freezing problem considered by Miller and Jiji [11] the flux distribution on the isothermal phase interface is taken from the Pohlhausen solution of the boundary layer equations. Inverse problems of potential flow involve a given streamline along which the pressure (or velocity) distribution is prescribed, as for example in the problems of free surface flow and flow through a nozzle which are described by Garabedian and Frank respectively.

8. Conclusions

Inverse free boundary problems which comprise Cauchy problems for the Laplace equation are encountered in various areas of engineering design. An inverted solution by analytic continuation as described herein provides an explicit expression for the geometry of physical boundaries of interest, thereby eliminating any need for numerical search or finding the inverse of an analytic function. The method can be generally applied by constructing a series approximation with correct asymptotic behavior to represent the given boundary along which Cauchy data is prescribed. These features of the present method are advantageous in formulating general design procedures like the one described herein for ECM tool design.

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REFERENCES

[1] D. E. Collett, R. C. Hewson-Browne and D. W. Windle, A Complex Variables Approach to Electrochemical Machining Problems, *J. Eng. Math.*, 4, 1 (1970) 24–37.
[2] H. Dietz, K. G. Gunther and K. Otto, Reproduction Accuracy with Electrochemical Machining: Determination of the Side Gap, *C.I.R.P.*, 22 (1973) 61–62.
[3] F. Fluerenbrock, R. D. Zerkle and J. F. Thorpe, Verification of a One-Dimensional Two-Phase Flow Model of the Frontal Gap in Electrochemical Machining, submitted to *ASME J. Eng. for Industry* (1974).
[4] L. S. Frank, Difference Methods for Solving the Improper Cauchy Problem Simulating Flow of a Perfect Gas Through a Nozzle, *Soviet Phys. Doklady*, 13, 9 (1969) 856–858.
[5] P. R. Garabedian, *Partial Differential Equations*, Wiley, New York (1964).
[6] P. R. Garabedian and H. M. Lieberstein, On the Numerical Calculation of Detached Shock Waves in Hypersonic Flow, *J. Aero Sciences*, 25, 2 (1958) 109–118.
[7] M. E. Goldstein and R. Siegel, Conformal Mapping for Heat Conduction in a Region with an Unknown Boundary, *Intl. J. Heat Mass Transfer*, 13 (1970) 1632–1636.
[8] K. Kawafune, T. Mikoshiba and K. Noto, Study on Shapes Processed by ECM, *C.I.R.P.*, 16 (1968) 345–352.
[9] A. L. Krylov, The Cauchy Problem for the Laplace Equation in the Theory of Electrochemical Metal Removal, *Soviet Phys. Doklady*, 13, 1 (1968) 15–17.
[10] S. Loutrel and N. Cook, Theoretical Model for High Rate Electrochemical Machining, *ASME Paper n73-Prod-2*, for meeting May 16 (1973).
[11] M. L. Miller and L. M. Jiji, Application of Point Matching Techniques to Two-Dimensional Solidification of Viscous Flow Over a Semi-Infinite Flate Plate, *ASME J. Appl. Mech.*, 92, 2 (1970) 508–517.
[12] R. Siegel, Shape of Porous Cooled Region for Surface Heat Flux and Temperature both Specified, *Intl. J. Heat Mass Transfer*, 16 (1973) 1807–1811.
[13] A. Thom and C. J. Apelt, *Field Computations in Engineering and Physics*, Van Nostrand, London (1961).
[14] J. F. Thorpe and R. D. Zerkle, Analytical Determination of the Equilibrium Electrode Gap in Electrochemical Machining, *Intl. J. Mach. Tool Des. Res.*, 9 (1969) 131–144.
[15] H. Tipton, The Determination of Tool Shape for ECM, *Machinery and Prod. Engr.*, Feb. (1968).